Mechanics and Relativity for Mathematicians - Wed Jan 31 2024

Write your name and student number on **all** sheets. There are four problems in this exam. You can earn 90 points in total. You are allowed to bring one (hand-written) two-sided A4 page as 'cheat sheet'.

PROBLEM 1: A Satellite Orbiting Around A Planet (all 8 points)
A satellite of mass m_1 is in an elliptical orbit around a planet of m_2 whose center is at F (one of the two foci of the ellipse) where $m_2 >> m_1$. P is the point of closest approach of the two masses and is a distance r_p from the foci F. The satellite's speed at the point P is v_p. The planet and the satellite are farthest apart at point A, $r_A = 5r_P$.

(a) Argue angular momentum and energy are conserved for such a system. [The gravitational force of attraction between two point masses is a central force, namely

$$\vec{F} = -\frac{dV(r)}{dr}\hat{r}\,. \tag{1}$$

Then, energy and angular momentum are conserved.(8pt)]

(b) Using conservation of angular momentum, find the speed of the satellite at point A in terms of v_P. [

$$L_A = m_1 r_A v_A (2pt) \tag{2}$$
$$L_P = m_1 r_P v_P (2pt) \tag{3}$$

By using conservation of angular momentum, $L_A = L_P$, we obtain $5v_A = v_P$ (4pt).]

(c) What is the energy of the system in terms of r_P? Is the energy of the satellite smaller or greater than zero? If you couldn't find the answer for part b, use $v_P = 3v_A$. Hint: First, using conservation of energy at A and P, find v_A in terms of r_P. [

$$E_A = E_P \tag{4}$$
$$E_A = \frac{1}{2}\frac{ml_A^2}{r_A^2} - G\frac{m_1 m_2}{r_A}(4pt) \tag{5}$$
$$\Rightarrow v_A = \left(\frac{1}{15}\frac{Gm_2}{r_p}\right)^{1/2}(2pt)\text{or for } v_P = 3v_A\,,\ v_A = \left(\frac{1}{5}\frac{Gm_2}{r_p}\right)^{1/2} \tag{6}$$

Pluging this in E,

$$E = -\frac{1}{6}\frac{Gm_1 m_2}{r_p} < 0(2pt) \tag{7}$$

This is what we expect from a bound orbit.]

(d) Should the satellite brake or fire its engine to change its orbit to hyperbola? Explain your answer briefly. [The energy of the satellite in a hyperbolic orbit is greater than in an elliptic orbit. Thus, it should fire its engine.(8pt)]

PROBLEM 2: Hinged Rod (all 8 points)
A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end. The rod is released from rest in the horizontal position. The moment of inertia of the rod about one of its ends is $I = 1/3ML^2$

(a) What is its angular velocity (magnitude and direction) when the rod reaches its lowest position? [

$$E_{initial} = E_{final}(2pt) \tag{8}$$
$$MgL = Mg\frac{1}{2}L + \frac{1}{2}I\omega^2(4pt) \tag{9}$$
$$Mg\frac{1}{2}L = \frac{1}{2}(\frac{1}{3}ML^2)\omega^2 \tag{10}$$

Thus, $\omega = \sqrt{\frac{3g}{L}}$. Let's imagine that the rod is hinged from its left end. Then, the angular velocity points into the page. (2pt)]

If we attach a mass M to the end of the rod and release the rod + mass M from rest in the horizontal position:

(b) How does the speed of the rod change when it reaches its lowest position? Does it increase, stay the same, or decrease? [The speed of the rod decreases because the moment of inertia increases. (8pt)]

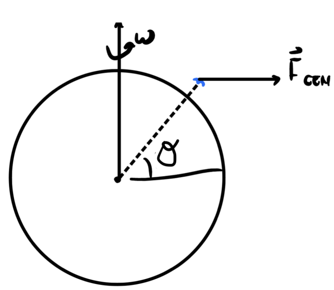
PROBLEM 3: A Bird Gliding (all 8 points)
The Earth rotates at a constant frequency around its axis, which generates two fictitious forces: the Coriolis force and the centrifugal force, given by

$$\vec{F}_{\text{Cor}} = -2m\,\vec{\omega} \times \dot{\vec{x}}\,, \qquad \vec{F}_{\text{centr}} = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{x})\,,$$

where $\vec{\omega}$ denotes the rotation velocity of the Earth.
Imagine that you are standing on the surface of the Earth. A bird of mass m is flying at velocity $\vec{v}$ in latitude θ in the northern hemisphere, heading due east. The bird soars to the east and maintains its height relative to the ground. Ignore air resistance and friction.

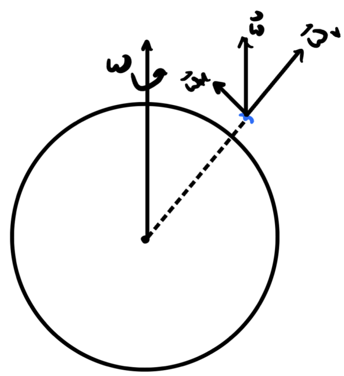
(a) Does the apparent gravitational force acting on the bird pass through the center of Earth? If not, what is the direction of the apparent gravity? Explain your answer briefly. Ignore the fictitious forces other than the centrifugal force. [

$$\begin{aligned}\vec{F}_{cent} &= -m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})\\ \vec{g}_{apparent} &= -g\hat{r} - m\,\vec{\omega} \times \vec{\omega} \times \vec{r}(2pt)\\ &= -g\hat{r} + m\omega^2 r\cos\theta(\text{to the right}(2pt))\end{aligned}$$



No, the apparent gravitational force passes under the center of Earth.(4pt)]

(b) Does the bird deflect to the west, east, north, or south? Which component of the Coriolis force is responsible for this deflection, namely vertical or horizontal? Explain your answer briefly. Ignore the fictitious forces other than the Coriolis force. [/We can decompose $\vec{\omega}$ as $\vec{\omega}_h$ and $\vec{\omega}_v$ as shown where $\vec{\omega} = \vec{\omega}_h + \vec{\omega}_v$.



$$\vec{F}^h_{cor} = -2\vec{\omega}_v \times \vec{v} \quad \text{and} \quad \vec{F}^v_{cor} = -2\vec{\omega}_h \times \vec{v}$$
$$F^h_{cor} = 2\omega \sin\theta v \quad \text{and} \quad F^v_{cor} = 2\omega \cos\theta v$$

The horizontal component causes the bird to head to the south (4pt). The deflection of the bird in the horizontal plane is caused by the horizontal component of the Coriolis force. The vertical component of the Coriolis force points to the center of Earth, and it works against the gravitational force using the right-hand rule.(4pt)]

PROBLEM 4: A Moving Clock (first two 8 points, last one 10 points)
A clock moving at velocity $u = 3c/5$ passes me, sitting at my origin, at $t = t' = 0$ according to it and my clock.

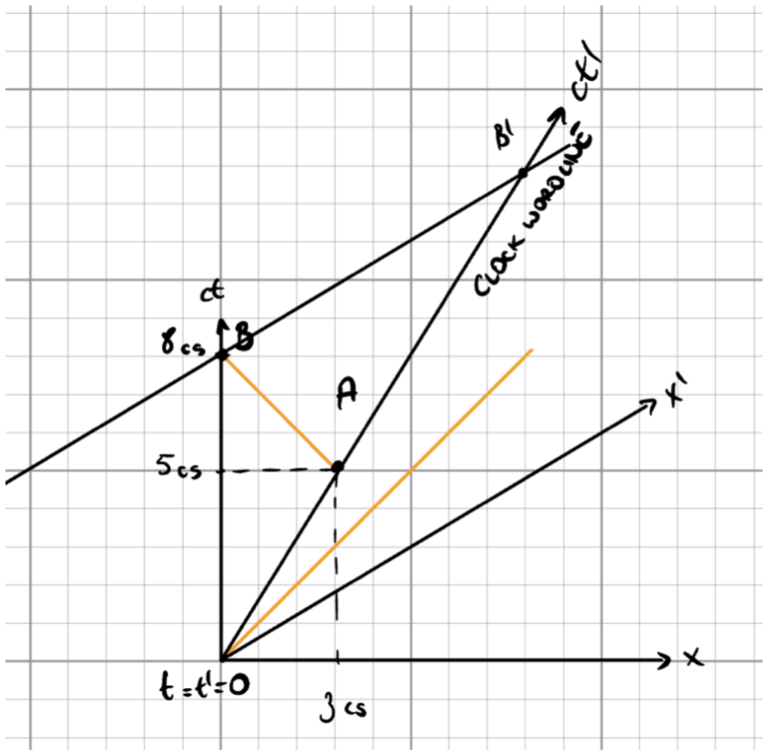
(a) What is its location in my frame when it ticks 4 seconds in its frame? [

$$\Delta t = \gamma \Delta t' = 5s(4pt) \tag{11}$$
$$d = v \cdot \Delta t = \frac{3}{5}c \cdot 5s = 3cs(4pt) \tag{12}$$

]

(b) If it emits a light pulse at that time, at what time t^* according to me will that pulse reach my origin? Use (ct, x) for me and (ct', x') for clock frame. [In my frame, the light is emitted from $x = 3cs$ and travels by c so it takes 3 seconds to receive it. At $t = 5 + 3 = 8s$, I received the light.(8pt)]

(c) Draw a spacetime diagram on the graph paper below. Indicate the worldline of the clock and the axes (ct, x) for me and (ct', x'). Label the events: Event A: the clock emits the light pulse. Event B: the pulse reaches me. Indicate on the diagram when



Event B happens in the clock frame. Is the time difference between these two events the same in both frames? Explain briefly.

[Correctly labelling A,B, B' (1pt) Correctly indicating the worldline of the clock and (ct, x), (ct', x')(1pt each) As the diagram indicates, the time interval in different inertial frames differs. This stems from the Lorentz transformations/postulates of special relativity. (1pt)]